Journal of Engineering Mathematics **31:** 379–395, 1997.

Low-frequency scattering of coated spherical obstacles

KIRIAKIE KIRIAKI, DEMOSTHENES POLYZOS and MARIOS VALAVANIDES
Department of Mathematics, National Technical University of Athens, Athens, Greece; [1]*Department of Mechanical Engineering,* [2]*Department of Chemical Engineering, University of Patras, Patras, Greece*

Received 30 May 1994; accepted in revised form 6 May 1996

Key words: Scattering theory, linear elasticity, low-frequency expansions, coated materials.

Abstract. This work deals with the scattering of a plane harmonic elastic wave by a penetrable spherical scatterer with a concentric spherical penetrable inclusion. We evaluate the zeroth and first-order approximations of the Rayleigh expansion of the displacement fields. The major line of applications belongs to the science of the particulate composite material. So, as an application of the method, a typical particulate composite material is examined and the behaviour of the scattering cross section with respect to the elastic properties of the medium is presented.

1. Introduction

The problem of scattering of a plane harmonic elastic wave by an obstacle appears as an exterior boundary-value problem for the time-reduced Navier equation with specific boundary conditions on the surface of the obstacle and prescribed asymptotic form in the neighborhood of infinity.

The general theory of elastic wave propagation is very well exposed by Kupradze [1]. The scattering of a longitudinal wave by a sphere was investigated for the first time by Ying and Truell [2]. Einspruch, Witterholt and Truell [3] have also solved the corresponding problem for transverse incidence. For the low-frequency region results are presented in [4]. *A posteriori* bounds to the error and estimates of the effects of changing the material constants and shape of the scatterer are described by Jones [5].

The scattering problem when the scatterer is a shell, is a rather complicated problem to be solved analytically. This is because one must apply the boundary conditions on a set of surfaces. The low-frequency theory for a penetrable elastic scatterer with an impenetrable core is presented in [6].

In this paper we examine the scattering by a penetrable elastic spherical scatterer with a concentric spherical penetrable core in the low-frequency region. More precisely, the purpose of this paper is to exploit the low-frequency scattering theory which is presented in [7] in order to evaluate the zeroth and first-order approximations of the Rayleigh expansion for the displacement fields. With the knowledge of these two terms we have enough information about the dynamic elastic fields.

The solution of this scattering problem, obtained for the low-frequency region, can be exploited in applications of the science of composite materials. A composite material can be considered as a homogeneous, isotropic elastic medium containing spherical inclusions. The modeling of these materials is of considerable engineering importance because other mechanical properties can be obtained from an analysis of their structure. By understanding the microscopic structure of a composite material we can extract general conclusions about their mechanical properties. So, the solution of the above-described scattering problem can be exploited in order to evaluate, with the aid of certain energy methods, the dynamic elastic moduli of the material [8,9]. For this reason the zeroth and first-order approximations of the displacement fields, as well as the scattering cross-section for real elastic materials which consist of a typical particulate composite, are given. Finally, we present numerical results from a particular composite material and its behaviour is discussed and explained. More precisely, the influence of elastic properties and size of "interphase" between the core (inclusion) and the exterior material (matrix) to scattering cross section is examined.

2. Statement of the problem

Assume a plane harmonic wave propagating in an infinite homogeneous isotropic elastic medium with Lamé constants λ_1, μ_1 and mass density ϱ_1. Inside this medium we consider a finite body, the scatterer with Lamé constants λ_2, μ_2 and mass density ϱ_2. We assume that the boundary of the scatterer is a smooth surface, say S_1. Let S_2 be the smooth boundary of another such body, the core, lying entirely within the scatterer S_1. We call V_1 the region exterior to S_1, V_2 the region between S_1 and S_2, and V_3 the region interior to S_2, filled by a medium with Lamé constants λ_3, μ_3 and mass density ϱ_3. The space V_2 is called the "interphase" of the scatterer and S_1, S_2, that is, the surfaces between the two layers, are the "interfaces".

We consider a longitudinal or a transverse incident plane wave, which, if we suppress the harmonic time dependence, takes the form

$$\mathbf{u}^{in}(\mathbf{r}) = \begin{cases} \hat{\mathbf{k}} \exp(ik_{p_1}\hat{\mathbf{k}} \cdot \mathbf{r}) \\ \hat{\mathbf{b}} \exp(ik_{s_1}\hat{\mathbf{k}} \cdot \mathbf{r}) \end{cases} \tag{1}$$

where $\hat{\mathbf{k}}$ is the propagation unit vector, $\hat{\mathbf{b}}$ is the polarization unit vector, $\hat{\mathbf{b}} \cdot \hat{\mathbf{k}} = 0$, and k_{p_1} and k_{s_1} are the wavenumbers of the P and S wave, respectively, in V_1.

The displacement field $\mathbf{v}(\mathbf{r})$ in the elastic medium, upon suppression of the time dependence $\exp\{-i\omega t\}$, satisfies the time-reduced Navier equation of dynamic elasticity

$$\Delta^* \mathbf{v} + \omega^2 \mathbf{v} = \mathbf{0}, \tag{2}$$

where

$$\Delta^* = c_s^2 \Delta + [c_p^2 - c_s^2]\nabla\nabla., \quad c_s^2 = \mu/\varrho, \quad c_p^2 = \lambda + 2\mu/\varrho \tag{3}$$

and λ, μ are the Lamé constants of the medium, ϱ is the mass density, ω is the angular frequency and c_p and c_s are the phase velocities of the shear and compressional wave respectively. In (2) it has been assumed that there are no body forces.

We assume that $\boldsymbol{\Psi}^{(1)}(\mathbf{r})$ is the displacement field outside $S_1, \boldsymbol{\Psi}^{(2)}(\mathbf{r})$ is the displacement field between S_1 nd S_2 and $\boldsymbol{\Psi}^{(3)}(\mathbf{r})$ the corresponding displacement field in the space V_3. The field $\boldsymbol{\Psi}^{(1)}(\mathbf{r})$ is the superposition of the incident field $\mathbf{u}^{in}(\mathbf{r})$ and the scattered field $\mathbf{u}(\mathbf{r})$, that is,

$$\boldsymbol{\Psi}^{(1)}(\mathbf{r}) = \mathbf{u}^{in}(\mathbf{r}) + \mathbf{u}(\mathbf{r}). \tag{4}$$

The displacement fields $\boldsymbol{\Psi}^{(j)}(\mathbf{r}), j = 1, 2, 3$ satisfy Equation (2). The scattered field $\mathbf{u}(\mathbf{r})$ satisfies the radiation conditions due to Kupradze [1]

$$\lim_{r\to+\infty} \mathbf{u}^p(\mathbf{r}) = \mathbf{0}, \quad \lim_{r\to+\infty} r\left\{\frac{\partial \mathbf{u}^p(\mathbf{r})}{\partial r} - ik_{p_1}\mathbf{u}^p(\mathbf{r})\right\} = \mathbf{0},$$

$$\lim_{r\to+\infty} \mathbf{u}^s(\mathbf{r}) = \mathbf{0}, \quad \lim_{r\to+\infty} r\left\{\frac{\partial \mathbf{u}^s(\mathbf{r})}{\partial r} - ik_{s_1}\mathbf{u}^s(\mathbf{r})\right\} = \mathbf{0}, \tag{5}$$

uniformly over all directions. The vectors $\mathbf{u}^p(\mathbf{r})$ and $\mathbf{u}^s(\mathbf{r})$ are the longitudinal and the transverse components of the scattered field, respectively.

The boundary conditions on the boundaries S_1 and S_2 are

$$\left.\begin{array}{l}\boldsymbol{\Psi}^{(k)}(\mathbf{r}') = \boldsymbol{\Psi}^{(k+1)}(\mathbf{r}') \\ T^{(k)}\boldsymbol{\Psi}^{(k)}(\mathbf{r}') = T^{(k+1)}\boldsymbol{\Psi}^{(k+1)}(\mathbf{r}')\end{array}\right\} \mathbf{r}' \in S_k \quad k = 1, 2, \tag{6}$$

where

$$T^{(j)} = 2\mu_j \hat{\boldsymbol{\eta}}' \cdot \nabla + \lambda_j \hat{\boldsymbol{\eta}}' \nabla \cdot + \mu_j \hat{\boldsymbol{\eta}}' \times \nabla, \quad j = 1, 2, 3 \tag{7}$$

is the surface-stress operator and $\hat{\boldsymbol{\eta}}$ is the unit normal on surfaces with direction from V_{j+1} to $V_j, j = 1, 2$.

The scattering problem consists in finding the fields $\boldsymbol{\Psi}^{(j)}(\mathbf{r}) j = 1, 2, 3$ which satisfy Equation (2), and the boundary conditions given by Equations (6), while the scattered field outside S_1 must satisfy radiation conditions given by (5).

In order to have the integral formulation of the problem, we will follow the ≪direct method≫ using Betti's formulae, as in [6]. The integral representation for the exterior total field is

$$\begin{aligned}\boldsymbol{\Psi}^{(1)}(\mathbf{r}) \;=\; & \frac{1}{4\pi\varrho_1}\sum_{k=1}^{2}\left\{\varrho_{k+1}\int_{V_{k+1}} \boldsymbol{\Psi}^{(k+1)}(\mathbf{r}') \cdot \left[\omega^2\left(1 - \frac{c^2_{s_{k+1}}}{c^2_{s_1}}\right) \cdot \tilde{\Gamma}^{(1)}(\mathbf{r}, \mathbf{r}')\right.\right. \\ & + \left.\left[c^2_{s_1}\left(c^2_{p_{k+1}} - c^2_{s_{k+1}}\right) - \frac{c^2_{s_{k+1}}}{c^2_{s_1}}\left(c^2_{p_1} - c^2_{s_1}\right)\right] \cdot \nabla\nabla \cdot \tilde{\Gamma}^{(1)}(\mathbf{r}, \mathbf{r}')\right] \mathrm{d}V(\mathbf{r}') \\ & \left. + \int_{s_k} \boldsymbol{\Psi}^{(k+1)}(\mathbf{r}') \cdot \left[T^{(k)} - T^{(k+1)}\right] \tilde{\Gamma}^{(1)}(\mathbf{r}, \mathbf{r}')\, \mathrm{d}S(\mathbf{r}')\right\} + \mathbf{u}^{in}(\mathbf{r}),\end{aligned} \tag{8}$$

where $\tilde{\Gamma}^{(1)}(\mathbf{r}, \mathbf{r}')$ is the fundamental dyadic solution in V_1 [4].

By using asymptotic analysis as described in [4] we may transform the integral representation (8), for large r, as follows:

$$\boldsymbol{\Psi}^{(1)}(\mathbf{r}) - \mathbf{u}^{in} = g_r(\hat{\mathbf{r}}, \hat{\mathbf{k}})\hat{\mathbf{r}} h(k_p r) + g_\theta(\hat{\mathbf{r}}, \hat{\mathbf{k}})\hat{\boldsymbol{\theta}} h(k_s r) + g_\phi(\hat{\mathbf{r}}, \hat{\mathbf{k}})\hat{\boldsymbol{\phi}} h(k_s r) + O\left(\frac{1}{r^2}\right), \tag{9}$$

where $\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}$ are the unit vectors of a spherical coordinate system having its origin interior to the scatterer. Further, $h(x) = \exp(ix)/ix$ is the zeroth order spherical Hankel function of the first kind and g_r, g_θ, g_ϕ are the radial and tangential scattering amplitudes given by the relations

$$\begin{aligned}g_r(\hat{\mathbf{r}}, \hat{\mathbf{k}}) \;=\; & \sum_{k=1}^{2}\left\{ik^3_{p_1}\left(1 - \frac{c^2_{p_{k+1}}}{c^2_{p_1}}\right)\left(\mathbf{q}_p^{(k+1)} \cdot \hat{\mathbf{r}}\right)\right. \\ & \left. + k^2_{p_1}\tilde{\mathbf{H}}_p^{(k+1)} : \left(\frac{\lambda_k - \lambda_{k+1}}{c^2_{p_1}}\,\tilde{\mathbf{I}} + 2\frac{\mu_k - \mu_{k+1}}{c^2_{p_1}}\hat{\mathbf{r}} \otimes \hat{\mathbf{r}}\right)\right\},\end{aligned} \tag{10}$$

$$g_\theta(\hat{\mathbf{r}},\hat{\mathbf{k}}) = \sum_{k=1}^{2}\left\{ik_{s_1}^3\left(1-\frac{c_{s_{k+1}}^2}{c_{s_1}^2}\right)(\mathbf{q}_s^{(k+1)}\cdot\hat{\boldsymbol{\theta}}) + k_{s_1}^2\frac{\mu_k-\mu_{k+1}}{c_{s_1}^2}[2\tilde{\mathbf{H}}_s^{(k+1)}:(\hat{\mathbf{r}}\otimes\hat{\boldsymbol{\theta}})+\mathbf{h}_s^{(k+1)}\cdot\hat{\boldsymbol{\phi}}]\right\}, \tag{11}$$

$$g_\phi(\hat{\mathbf{r}},\hat{\mathbf{k}}) = \sum_{k=1}^{2}\left\{ik_{s_1}^3\left(1-\frac{c_{s_{k+1}}^2}{c_{s_1}^2}\right)(\mathbf{q}_s^{(k+1)}\cdot\hat{\boldsymbol{\phi}}) + k_{s_1}^2\frac{\mu_k-\mu_{k+1}}{c_{s_1}^2}[2\tilde{\mathbf{H}}_s^{(k+1)}:(\hat{\mathbf{r}}\otimes\hat{\boldsymbol{\phi}})+\mathbf{h}_s^{(k+1)}\cdot\hat{\boldsymbol{\theta}}]\right\}, \tag{12}$$

The inner double product is defined as $(\mathbf{a}\otimes\mathbf{b}):(\mathbf{c}\otimes\mathbf{d})=(\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$. The quantities in (10)–(12) are given by the relations

$$\tilde{\mathbf{H}}_\tau^{(k)} = \frac{1}{4\pi\varrho_1}\int_{S_{k-1}}\boldsymbol{\Psi}^{(k)}(\mathbf{r}')\otimes\hat{\boldsymbol{\eta}}'\exp(-ik_{\tau_1}\hat{\mathbf{r}}\cdot\mathbf{r}')\,\mathrm{d}S(\mathbf{r}'), \tag{13}$$

$$\mathbf{q}_\tau^{(k)} = \frac{\varrho_k}{4\pi\varrho_1}\int_{V_k}\boldsymbol{\Psi}^{(k)}(\mathbf{r}')\exp(-ik_{\tau_1}\hat{\mathbf{r}}\cdot\mathbf{r}')\,\mathrm{d}V(\mathbf{r}'), \tag{14}$$

$$\mathbf{h}_\tau^{(k)} = \frac{1}{4\pi\varrho_1}\int_{S_{k-1}}\boldsymbol{\Psi}^{(k)}(\mathbf{r}')\times\hat{\boldsymbol{\eta}}'\exp(-ik_{\tau_1}\hat{\mathbf{r}}\cdot\mathbf{r}')\,\mathrm{d}S(\mathbf{r}'), \tag{15}$$

where in (13)–(15) τ is p or s.

In [4] it has been proved, that the scattering cross section σ^p corresponding to an incident P wave is given by the relation

$$\sigma^p = k_{p_1}\int_{|\hat{\mathbf{r}}|=1}\left\{k_{p_1}^{-3}|g_r(\hat{\mathbf{r}},\hat{\mathbf{k}})|^2 + k_{s_1}^{-3}(|g_\theta(\hat{\mathbf{r}},\hat{\mathbf{k}})|^2+|g_\phi(\hat{\mathbf{r}},\hat{\mathbf{k}})|^2)\right\}\,\mathrm{d}\Omega, \tag{16}$$

where the integration is taken over the unit sphere. Substituting in (16) the relations for the scattering amplitudes given by the Equations (10)–(12), we obtain the scattering cross section for P-incidence.

The scattering cross section for S-incidence is written as σ^s and is given by a formula similar to that of the P-incidence, where all the quantities are evaluated for the case of S-incidence.

3. The problem in the low-frequency region

The displacement fields $\boldsymbol{\Psi}^{(j)}(\mathbf{r}), j=1,2,3$ considered as functions of the wave number k_s or k_p are analytic in a neighborhood of zero [5]. Consequently, they can be expanded in a convergent power series of the wave number k_s or k_p. So, we have

$$\boldsymbol{\Psi}^{(j)}(\mathbf{r}) = \sum_{n=0}^{\infty}\frac{(ik_{p_1})}{n!}\boldsymbol{\Phi}_n^{(j)}(\mathbf{r}) = \sum_{n=0}^{\infty}\frac{(i\tau_1 k_1)^n}{n!}\boldsymbol{\Phi}_n^{(j)}(\mathbf{r}), \quad \mathbf{r}\in V_j \quad j=1,2,3, \tag{17}$$

where $k_1 = k_{s_1}$ is the wavenumber of the transverse wave in the space V_1 and

$$\tau_j^2 = \frac{c_{s_j}^2}{c_{p_j}^2} = \frac{\mu_j}{\lambda_j+2\mu_j}, \quad j=1,2,3. \tag{18}$$

Substituting (17) in (2), we conclude that the low-frequency coefficients $\boldsymbol{\Phi}_n^{(j)}(\mathbf{r})$ satisfy the equations

$$\tau_j^2 \Delta \boldsymbol{\Phi}_n^{(j)}(\mathbf{r}) + (1-\tau_j^2)\nabla\nabla\cdot\boldsymbol{\Phi}_n^{(j)}(\mathbf{r}) - n(n-1)q_j \boldsymbol{\Phi}_{n-2}^{(j)}(\mathbf{r}) = \mathbf{0}, \quad n = 0,1,2,\ldots, \tag{19}$$

where

$$q_j = \frac{c_{p_1}^2}{c_{p_j}^2}, \quad j = 1,2,3. \tag{20}$$

Note that for $n = 0, 1$ the last term on the left-hand side of (19) vanishes.

The boundary conditions are transformed in the low-frequency region into the boundary conditions

$$\left.\begin{aligned} &\boldsymbol{\Phi}_n^{(k)}(\mathbf{r}') = \boldsymbol{\Phi}_n^{(k+1)}(\mathbf{r}'), \\ &T^{(k)}\boldsymbol{\Phi}_n^{(k)}(\mathbf{r}') = T^{(k+1)}\boldsymbol{\Phi}_n^{(k+1)}(\mathbf{r}'). \end{aligned}\right\} \quad (\mathbf{r}' \in S_k, \quad k = 1,2) \tag{21}$$

We derive the integral representations of $\boldsymbol{\Phi}_n^{(j)}$ by substituting in (8) the low-frequency expansions of all the quantities that appear in it and equating the equal powers of k_1. So, we conclude that

$$\begin{aligned} l_j \boldsymbol{\Phi}_n^{(j)}(\mathbf{r}) \ =\ & \frac{1}{4\pi\mu_1}\sum_{\varrho=0}^{n}\binom{n}{\varrho}\left\{\sum_{k=1}^{2}\left[\varrho_{k+1}\int_{V_{k+1}}\boldsymbol{\Phi}_\varrho^{(k+1)}(\mathbf{r}')\cdot\right.\right. \\ & \cdot\left[\frac{(c_{s_1}^2 - c_{s_{k+1}}^2)(n-\varrho)(n-\varrho-1)}{\tau_1^2}\tilde{V}_{n-\varrho-2}^{(1)}(\mathbf{r},\mathbf{r}')\right. \\ & \left. + \frac{c_{s_1}^2(c_{p_{k+1}}^2 - c_{s_{k+1}}^2) - c_{s_{k+1}}^2(c_{p_1}^2 - c_{s_1}^2)}{c_{s_1}^2}\nabla_{\mathbf{r}'}\nabla_{\mathbf{r}'}\tilde{V}_{n-\varrho}^{(1)}(\mathbf{r},\mathbf{r}')\right]\,\mathrm{d}V(\mathbf{r}') \\ & \left.\left. + \int_{S_k}\boldsymbol{\Phi}_\varrho^{(k+1)}(\mathbf{r}')\cdot(T^{(k)} - T^{(k+1)})\tilde{V}_{n-\varrho}^{(1)}(\mathbf{r},\mathbf{r}')\,\mathrm{d}S(\mathbf{r}')\right] + \mathbf{a}_n(\hat{\mathbf{k}}\cdot\mathbf{r})^n\right\} \\ & (n = 0,1,\ldots, \quad k = 1,2, \quad j = 1,2,3), \end{aligned} \tag{22}$$

where

$$\mathbf{a}_n = \begin{cases} \hat{\mathbf{k}}, & \text{for } P\text{-incidence} \\ \dfrac{\hat{\mathbf{b}}}{\tau_1^n}, & \text{for } S\text{-incidence} \end{cases} \tag{23}$$

$$l_j = \frac{1}{3}\left(2\frac{\mu_j}{\mu_j} + \frac{\lambda_j + 2\mu_j}{\lambda_1 + 2\mu_1}\right), \quad j = 1,2,3, \tag{24}$$

and $\tilde{V}_n^{(1)}$ is the n-th term of the expansion of the fundamental solution $\tilde{\Gamma}^{(1)}(\mathbf{r},\mathbf{r}')$ at low frequency given by the relation [4]

$$\tilde{V}_n^{(1)}(\mathbf{r},\mathbf{r}') = \frac{|\mathbf{r}-\mathbf{r}'|^{n-1}}{\tau_1^n}\left\{\left(1 + \frac{\tau_1^{n+2}-1}{n+2}\right)\tilde{I} + (n-1)\frac{\tau_1^{n+2}-1}{n+2}\frac{(\mathbf{r}-\mathbf{r}')\otimes(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^2}\right\}. \tag{25}$$

In order to derive the low-frequency expansion for the scattering amplitudes, we will insert in (10)–(12) the Rayleigh expansions of all the quantities which appear in them. So, taking into account the low-frequency approximations of the scattering amplitudes, we can use (16) in order to evaluate the leading-term approximation for the scattering cross-section. So, we have

For P-incidence

$$
\begin{aligned}
\sigma^p = & \left\{ \frac{\tau_1(\tau_1^3+2)}{12\pi}\left[\left(\frac{\varrho_2}{\varrho_1}-1\right)V_2+\left(\frac{\varrho_3}{\varrho_1}-1\right)\right]+\frac{\tau_1^3}{60\pi}\left[15\tau_1(1-2\tau_1^2)^2\left(\frac{\lambda_2-\lambda_1}{\lambda_1}\right)^2\right.\right. \\
& +4(\tau_1^5-1)\left(\frac{\mu_2-\mu_1}{\mu_1}\right)^2+20\tau_1^3(1-2\tau_1^2)\left.\left(\frac{\lambda_2-\lambda_1}{\lambda_1}\right)\left(\frac{\mu_2-\mu_1}{\mu_1}\right)\right]\cdot(\mathrm{tr}\,\tilde{\mathbb{A}}_2)^2 \\
& +\frac{\tau_1^3}{60\pi}\left[15\tau_1(1-2\tau_1^2)^2\left(\frac{\lambda_3-\lambda_2}{\lambda_1}\right)^2+4(\tau_1^5-1)\left(\frac{\mu_3-\mu_2}{\mu_1}\right)^2+20\tau_1^3(1-2\tau_1^2)\right. \\
& \cdot\left.\left(\frac{\lambda_3-\lambda_2}{\lambda_1}\right)\left(\frac{\mu_3-\mu_2}{\mu_1}\right)\right]\cdot(\mathrm{tr}\,\tilde{\mathbb{A}}_3)^2 \\
& +\frac{\tau_1^3}{30\pi}\left[15\tau_1(1-2\tau_1^2)^2\left(\frac{\lambda_2-\lambda_1}{\lambda_1}\right)\left(\frac{\lambda_3-\lambda_2}{\lambda_1}\right)+4(\tau_1^5-1)\left(\frac{\mu_2-\mu_1}{\mu_1}\right)\left(\frac{\mu_3-\mu_2}{\mu_1}\right)\right. \\
& +10\tau_1^3(1-2\tau_1^2)\left.\left(\frac{\lambda_2-\lambda_1}{\lambda_1}\frac{\mu_3-\mu_2}{\mu_1}+\frac{\lambda_3-\lambda_2}{\lambda_1}\frac{\mu_2-\mu_1}{\mu_1}\right)\right](\mathrm{tr}\,\tilde{\mathbb{A}}_2)(\mathrm{tr}\,\tilde{\mathbb{A}}_3) \\
& +\frac{\tau_1^3}{15\pi}(\tau_1^5-1)\left(\frac{\mu_2-\mu_1}{\mu_1}\right)^2\tilde{\mathbb{A}}_2:\tilde{\mathbb{A}}_2+\frac{\tau_1^3}{15\pi}(\tau_1^5+4)\left(\frac{\mu_2-\mu_1}{\mu_1}\right)^2\tilde{\mathbb{A}}_2:\tilde{\mathbb{A}}_2^T \\
& +\frac{\tau_1^3}{15\pi}(\tau_1^5-1)\left(\frac{\mu_3-\mu_2}{\mu_1}\right)^2\tilde{\mathbb{A}}_3:\tilde{\mathbb{A}}_3+\frac{\tau_1^3}{15\pi}(\tau_1^5+4)\left(\frac{\mu_3-\mu_2}{\mu_1}\right)^2\tilde{\mathbb{A}}_3:\tilde{\mathbb{A}}_3^T \\
& +\frac{2\tau_1^3}{15\pi}(\tau_1^5-1)\left(\frac{\mu_2-\mu_1}{\mu_1}\right)\left(\frac{\mu_3-\mu_2}{\mu_1}\right)\tilde{\mathbb{A}}_2:\tilde{\mathbb{A}}_3 \\
& +\frac{2\tau_1^3}{15\pi}(\tau_1^4+4)\left(\frac{\mu_2-\mu_1}{\mu_1}\right)\left(\frac{\mu_3-\mu_2}{\mu_1}\right)\tilde{\mathbb{A}}_2:\tilde{\mathbb{A}}_3^T \\
& -\frac{\tau_1^3}{6\pi}\left(\frac{\mu_2-\mu_1}{\mu_1}\right)^2|\boldsymbol{\alpha}_2|^2-\frac{\tau_1^3}{6\pi}\left(\frac{\mu_3-\mu_2}{\mu_1}\right)^2|\boldsymbol{\alpha}_3|^2 \\
& \left.-\frac{\tau_1^3}{3\pi}\left(\frac{\mu_2-\mu_1}{\mu_1}\right)\left(\frac{\mu_3-\mu_2}{\mu_1}\right)\boldsymbol{\alpha}_2\cdot\boldsymbol{\alpha}_3\right\}k_1^4+O(k_1^6),
\end{aligned}
\tag{26}
$$

where

$$
\tilde{\mathbb{A}}_2=\int_{S_1}\boldsymbol{\Phi}_1^{(2)}(\mathbf{r}')\otimes\hat{\mathbf{n}}'\,\mathrm{d}S(\mathbf{r}'),
\tag{27}
$$

$$
\tilde{\mathbb{A}}_3=\int_{S_2}\boldsymbol{\Phi}_1^{(3)}(\mathbf{r}')\otimes\hat{\mathbf{n}}'\,\mathrm{d}S(\mathbf{r}'),
\tag{28}
$$

and $\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3$ are the vector invariants of $\tilde{\mathbb{A}}_2,\tilde{\mathbb{A}}_3$ respectively. Similarly, we can obtain the leading-term approximation for the S-incidence.

4. Scattering by a penetrable spherical scatterer with a concentric spherical penetrable core

Consider a spherical scatterer of radius b, centered at the origin of the coordinate system, which contains a concentric spherical inclusion of radius a. As we have already seen, the scattering problem, using low-frequency techniques, is reduced to a sequence of problems governed by (19) and satisfying the boundary conditions given by (21). In order to solve these problems we can exploit the fact that the integral representation for the nth-order coefficient given by (22), provides a particular solution of the nonhomogeneous equation (19). So, we only need to solve the corresponding homogeneous equation. Introducing the Papkovich potentials we can evaluate the solution of the above problem. The details of this method can be found in [6]. Based on this method, after long and tedious manipulations, we have the following representations for the displacement fields:

For the zeroth order approximation:

$$\boldsymbol{\Phi}_0^{(1)}(\mathbf{r}) = \boldsymbol{\Phi}_0^{(2)}(\mathbf{r}) = \boldsymbol{\Phi}_0^{(3)} = \mathbf{a}_0, \tag{29}$$

For the first order approximation:

$$\begin{aligned}\boldsymbol{\Phi}_1^{(j)}(\mathbf{r}) \;=\; & \left[s_0^{(j)} + q_0^{(j)}\left(\frac{r}{a}\right)^2 + p_0^{(j)}\left(\frac{a}{r}\right)^3 + t_0^{(j)}\left(\frac{a}{r}\right)^5\right](\mathbf{a}_1\cdot\hat{\mathbf{k}})\mathbf{r}\\ & + \left[s_{Ak}^{(j)} + q_{Ak}^{(j)}\left(\frac{r}{a}\right)^2 + p_{Ak}^{(j)}\left(\frac{a}{r}\right)^3 + t_{Ak}^{(j)}\left(\frac{a}{r}\right)^5\right](\mathbf{a}_1\otimes\hat{\mathbf{k}})\cdot\mathbf{r}\\ & + \left[s_{kA}^{(j)} + q_{kA}^{(j)}\left(\frac{r}{a}\right)^2 + p_{kA}^{(j)}\left(\frac{a}{r}\right)^3 + t_{kA}^{(j)}\left(\frac{a}{r}\right)^5\right](\hat{\mathbf{k}}\otimes\mathbf{a}_1)\cdot\mathbf{r}\\ & + \left[s^{(j)} + q^{(j)}\left(\frac{r}{a}\right)^2 + p^{(j)}\left(\frac{a}{r}\right)^3 + t^{(j)}\left(\frac{a}{r}\right)^5\right](\mathbf{a}_1\otimes\hat{\mathbf{k}}):(\hat{\mathbf{r}}\otimes\hat{\mathbf{r}})\mathbf{r},\\ & \text{for } j = 1,2,3,\end{aligned} \tag{30}$$

where all the coefficients which appear in (30) are given in Appendix A.

5. An application to a typical particulated composite material

In what follows we will use the results given by Equation (30) in order to evaluate the first-order low-frequency approximations of the displacement fields for a typical particulated composite material. We assume that V_1 (matrix) is filled with Epoxy ($E_1 = 5.033$ GPa, $\mu_1 = 1.798$ GPa, $\nu_1 = 0.4$, $\varrho_1 = 1261$ kg/m^3) and V_3 (inclusion) with Glass ($E_3 = 68.9$ GPa, $\mu_3 = 28.01$ GPa, $\nu_3 = 0.23$, $\varrho_3 = 2620$ kg/m^3) where E_1, E_3 are the Young's moduli, μ_1, μ_3 the shear moduli and ν_1, ν_3 the Poisson ratios of the materials.

The interphase is a material with suitable properties in order to match the actual behaviour of the two main phases of the composite (matrix-inclusion) ($E_2 = 48.173$ GPa, $\mu_2 = 17.45$ GPa, $\nu_2 = 0.38$, $\varrho_2 = 1387$ kg/m^3). We also assume that the ratio of the radii is $b/a = 1.2$.

The first approximation of the displacement fields is

$$\boldsymbol{\Phi}_1^{(1)}(\mathbf{r}) \;=\; 0.5675\left(\frac{a}{r}\right)^3(\mathbf{a}\cdot\hat{\mathbf{k}})\mathbf{r} - 0.1815\left(\frac{a}{r}\right)^3\left[\mathbf{a}_1\otimes\hat{\mathbf{k}} + \hat{\mathbf{k}}\otimes\mathbf{a}_1\right]\cdot\mathbf{r}$$

$$-2.722\left(\frac{a}{r}\right)^3(\mathbf{a}_1\otimes\hat{\mathbf{k}}):(\hat{\mathbf{r}}\otimes\hat{\mathbf{r}})\mathbf{r}-0.7722\left(\frac{a}{r}\right)^5(\mathbf{a}_1\cdot\hat{\mathbf{k}})\mathbf{r}$$
$$-0.7722\left(\frac{a}{r}\right)^5(\mathbf{a}_1\otimes\hat{\mathbf{k}}+\hat{\mathbf{k}}\otimes\mathbf{a}_1)\cdot\mathbf{r}+3.861\left(\frac{a}{r}\right)^5(\mathbf{a}_1\otimes\hat{\mathbf{k}}:\hat{\mathbf{r}}\otimes\hat{\mathbf{r}})\mathbf{r}, \quad (31)$$

$$\begin{aligned}\boldsymbol{\Phi}_1^{(2)}(\mathbf{r}) = {} & \left[-0.3475-2.091\left(\frac{r}{a}\right)^2+45.83\left(\frac{a}{r}\right)^3\right. \\ & \left.-12.604\left(\frac{a}{r}\right)^5\right]\times 10^{-3}(\mathbf{a}_1\cdot\hat{\mathbf{k}})\mathbf{r}+\left[581.8+7.537\left(\frac{r}{a}\right)^2\right. \\ & \left.-5.049\left(\frac{a}{r}\right)^3-12.60\left(\frac{a}{r}\right)^5\right]\times 10^{-3}(\mathbf{a}_1\otimes\hat{\mathbf{k}})\cdot\mathbf{r}+[-418.2 \\ & +7.537\left(\frac{r}{a}\right)^2-5.049\left(\frac{a}{r}\right)^3-12.604\left(\frac{a}{r}\right)^5\Big]\times 10^{-3}(\hat{\mathbf{k}}\otimes\mathbf{a}_1)\cdot\mathbf{r} \\ & +\left[-8.802\left(\frac{r}{a}\right)^2-63.12\left(\frac{a}{r}\right)^3+63.02\left(\frac{a}{r}\right)^5\right]\times 10^{-3} \\ & (\mathbf{a}_1\otimes\hat{\mathbf{k}}):(\hat{\mathbf{r}}\otimes\hat{\mathbf{r}})\mathbf{r}, \end{aligned} \quad (32)$$

$$\begin{aligned}\boldsymbol{\Phi}_1^{(3)}(\mathbf{r}) = {} & \left[31.66-0.8713\left(\frac{r}{a}\right)^2\right]\times 10^{-3}(\mathbf{a}_1\cdot\hat{\mathbf{k}})\mathbf{r} \\ & +\left[565.9+5.758\left(\frac{r}{a}\right)^2\right]\times 10^{-3}(\mathbf{a}_1\otimes\hat{\mathbf{k}})\cdot\mathbf{r} \\ & +\left[-434.1+5.758\left(\frac{r}{a}\right)^2\right]\times 10^{-3}(\hat{\mathbf{k}}\otimes\mathbf{a}_1)\cdot\mathbf{r} \\ & -8.902\times 10^{-3}\left(\frac{r}{a}\right)^2(\mathbf{a}_1\otimes\hat{\mathbf{k}}):(\hat{\mathbf{r}}\otimes\hat{\mathbf{r}})\mathbf{r}. \end{aligned} \quad (33)$$

At this point it should be noted that the interphasial coefficients of order $\mathrm{O}(r^3)$, *i.e.* $q_{Ak}^{(3)}, q_{kA}^{(3)}, q_0^{(3)}$ and $q^{(3)}$ of (30) have a special behavior as the ratio b/a runs from 1 to large values. This behavior is apparent in Figure 2 where the coefficients $q_{Ak}^{(3)} = q_{kA}^{(3)}, q_0^{(3)}$ and $q^{(3)}$ of the above-mentioned particulate composite are presented as a function of the ratio b/a. In this figure one can see that for large values of $b/a (b/a > 3)$ the terms of order $\mathrm{O}(r^3)$ become very small and appear to have a behavior similar to those corresponding to the value $b/a = 1$ (particle without interphase). Furthermore the four coefficients have a maximum at the same point $b/a = 1.2$. As an explanation for this behavior of the solution $\boldsymbol{\Phi}_1^{(3)}(\mathbf{r})$, we can say that the two interfaces S_1 and S_2, because of the displacement and stress continuity, cannot easily cooperate when the space between them is very thin, especially for a critical value of b/a. Thus, in this case terms of order $O(r^3)$ appear at the displacement field $\boldsymbol{\Phi}_1^{(3)}(\mathbf{r})$, in order ≪to help≫ the material components to satisfy the boundary conditions on the surfaces S_1 and S_2.

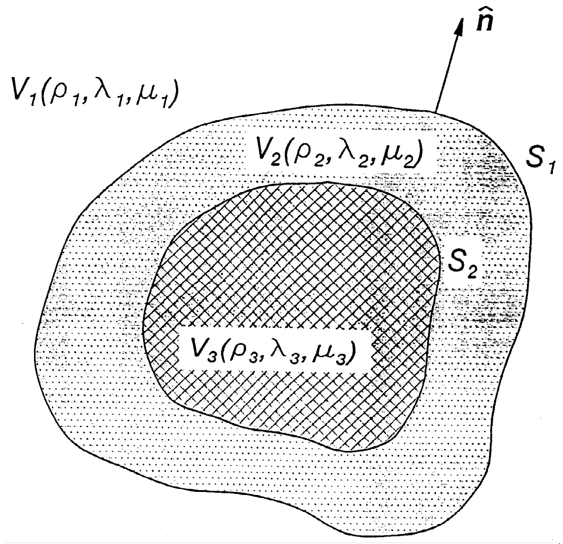

Figure 1. The geometry of the elastic scatterer with an elastic core.

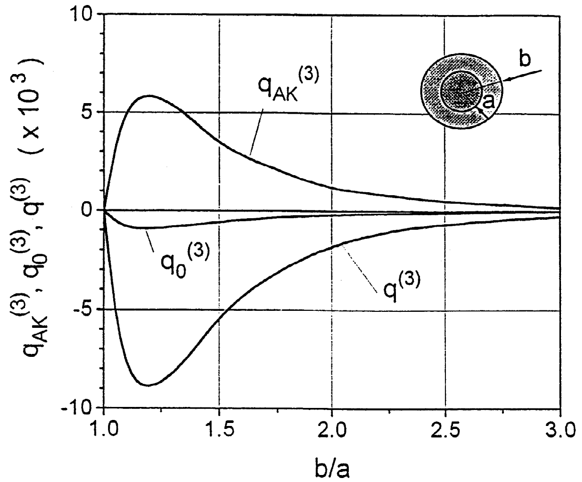

Figure 2. Effect of the interphase thickness on the interphasial coefficients of order $O(r^3)$ for a spherical particle composite material with material properties $\mu_1 = 1.798$ GPa, $\nu_1 = 0.4$, $\varrho_1 = 1261$ kg/m^3 for the matrix, $\mu_2 = 17.45$ GPa, $\nu_2 = 0.38$, $\varrho_2 = 1378$ kg/m^3 for the interphase and $\mu_3 = 28.01$ GPa, $\nu_3 = 0.23$, $\varrho_3 = 2620$ kg/m^3 for the inclusion.

6. The behavior of the leading term of scattering cross sections in low frequencies

As it is proposed in [8,9] a knowledge of the leading-term of the scattering cross section is needed in order to find the dynamical properties of the composite.

From (26) the following formulae for the scattering cross sections hold

$$\sigma^p = \bar{\sigma}^p V_3^2 k_1^4 + O(k_1^6), \tag{34}$$

$$\sigma^s = \bar{\sigma}^s V_3^2 k_1^4 + O(k_1^6). \tag{35}$$

In the sequel the dependence of the reduced leading terms $\bar{\sigma}^p, \bar{\sigma}^s$ on the relative elastic properties of the composite is examined for two special cases. First, for the spherical particle

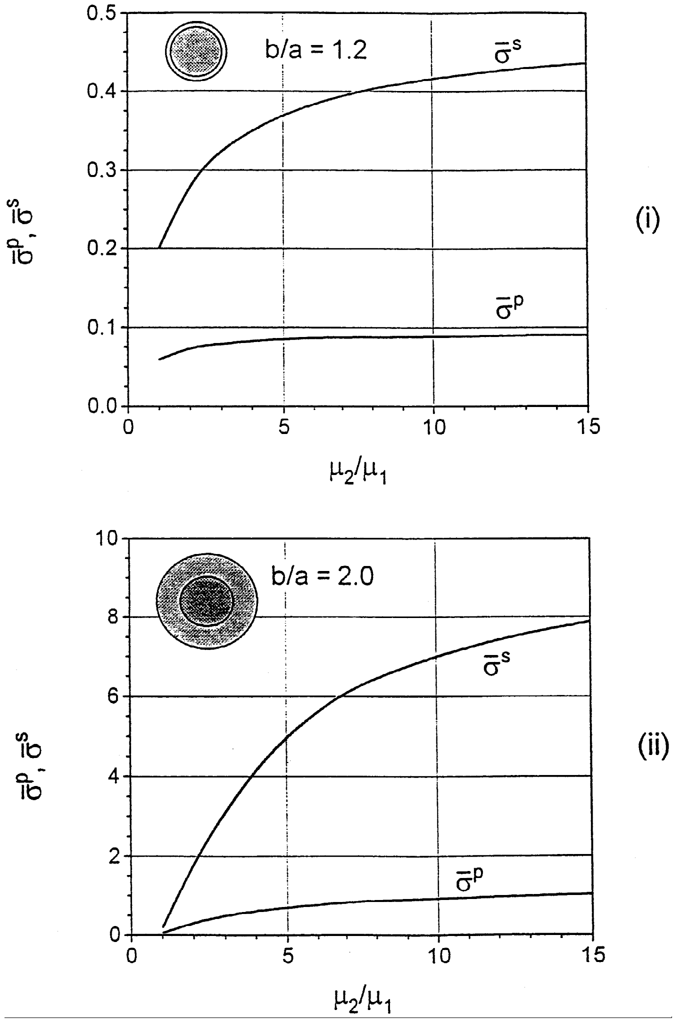

Figure 3. Effect of the interphase/matrix shear moduli ratio μ_2/μ_1, on the reduced energy scattering cross sections $\bar{\sigma}^p, \bar{\sigma}^s$ for a spherical particle composite material with material properties $\mu_1, \mu_3, \nu_1, \nu_2, \varrho_1, \varrho_2$ and ϱ_3 as referenced in Figure 2, for (i) $b/a = 1.2$ and (ii) $b/a = 2.0$.

composite material which is presented in the previous paragraph, the variation of $\bar{\sigma}^p, \bar{\sigma}^s$, for $b/a = 1.2$ and $b/a = 2.0$ is presented in Figures 3(i) and 3(ii), respectively, as the interphase matrix shear-moduli ratio μ_2/μ_1 takes values from 1 to 15. We can observe the strong dependence of $\bar{\sigma}^s$ on b/a and on μ_2/μ_1, whereas $\bar{\sigma}^p$ remains unaffected by an increase of μ_2/μ_1 above a certain value. This is due to the fact that μ_2/μ_1 expresses the ratio of the shear moduli and, consequently, is directly related to S-incidence.

The thickness of the interphase has a major effect on both $\bar{\sigma}^p$ and $\bar{\sigma}^s$. It extends the impact of μ_2/μ_1 on $\bar{\sigma}^p$ and $\bar{\sigma}^s$ over a larger area. The coupling "role" of the interphase is confirmed. The relative properties of the above composite material assume the values $\mu_3/\mu_1 = 15.6, \varrho_3/\varrho_1 = 2.077, \varrho_2/\varrho_1 = 1.1$.

Finally, the special case of a composite material with identical matrix inclusion and an interphase with the same Poisson ratio and density to the above, but different shear modulus, is considered. The dependence of $\bar{\sigma}^p$ and $\bar{\sigma}^s$ on μ_2/μ_1 for $b/a = 1.2$ and $b/a = 2.0$ is presented in Figures 4(i) and 4(ii), respectively.

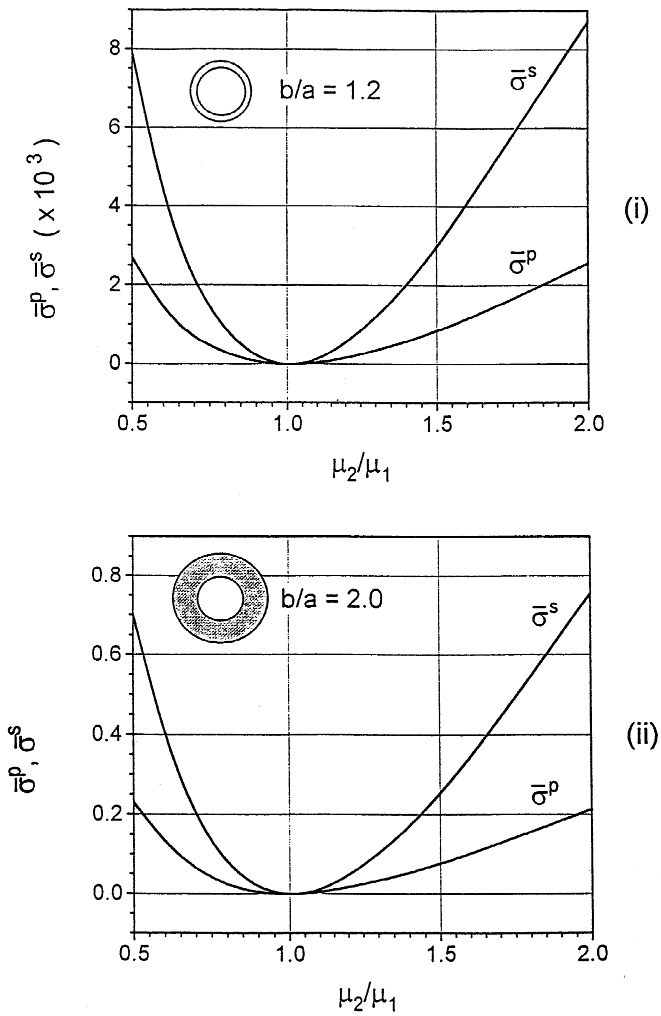

Figure 4. Effect of the interphase/matrix shear-moduli ratio μ_2/μ_1, on the reduced energy scattering cross sections $\bar{\sigma}^p, \bar{\sigma}^s$ for a spherical particle composite material with $\mu_1 = \mu_3$, $\nu_1 = \nu_2 = \nu_3$, and $\varrho_1 = \varrho_2 = \varrho_3$ for (i) $b/a = 1.2$. and (b) $b/a = 2.0$.

The trend of the curves is identical for $b/a = 1.2$ and $b/a = 2.0$ owing to the similarity of the matrix and inclusion. The effect of the interphase thickness is strong, as can be confirmed by the augmentation by two orders of magnitude of $\bar{\sigma}^p$ and $\bar{\sigma}^s$ when b/a increases from 1.2 to 2.0. In addition, the dependence of $\bar{\sigma}^p$ and $\bar{\sigma}^s$ on μ_2/μ_1 is very pronounced.

7. Discussion

The major line of applications of the present work belongs to the science of the particulate composite materials. A typical particulate composite material is usually a homogeneous isotropic elastic medium (matrix), containing elastic particles (inclusions) surrounded by an elastic interphase due to the imperfect adhesion between matrix and particles. A mathematical problem which has a considerable engineering importance in this area is the evaluation of the dynamic properties of a particulate composite. Almost all the works that have appeared in the literature for this purpose are based on the same method. This method, consists in solving first the scattering problem in a microscopic level (matrix-inclusion) and, from relations of conservation of energy, the macroscopic dynamic properties of the composite are evaluated. The same method has been followed in [8,9] where the macroscopic dynamic properties of the composite have been evaluated through the low-frequency leading term of the scattering cross section which gives a measure of the total energy scattered by the particles. So, the solution of the examined scattering problem in this work can be exploited in order to evaluate the dynamic elastic moduli of a composite at a macroscopic level.

Appendix A

$$q_0^{(1)}=q_{Ak}^{(1)}=q_{kA}^{(1)}=q^{(1)}=p_0^{(3)}=p_{Ak}^{(3)}=p_{kA}^{(3)}=p^{(3)}=0, \tag{A.1}$$

$$s_0^{(1)}=s_{kA}^{(1)}=s^{(1)}=s^{(2)}=t_0^{(3)}=t_{Ak}^{(3)}=t_{kA}^{(3)}=t^{(3)}=0, \tag{A.2}$$

$$s_{Ak}^{(1)}=1, \tag{A.3}$$

$$\begin{aligned} p_0^{(1)} = &\left[\frac{(2\mu_2+\mu_3)\lambda_2}{2\mu_1^2}\left(\frac{b}{a}\right)^3\left[1-\left(\frac{b}{a}\right)^2\right]\tau_2^2X_1-\frac{3}{2}\frac{\lambda_3\mu_3}{\mu_1^2}\left(\frac{b}{a}\right)^3\tau_3^2Y_1\right.\\ &+\frac{3}{5}\frac{\mu_2(\mu_2-\mu_3)}{\mu_1^2}(\tau_2^2-1)\left(\frac{b}{a}\right)^3\left[1-\left(\frac{b}{a}\right)^2\right]X_3\\ &+\frac{1}{5}\left[\left[\frac{\mu_2-\mu_3}{2\mu_1}+\frac{2\mu_2+\mu_3}{2\mu_1}\left(\frac{b}{a}\right)^3\right]\right.\\ &\left.\cdot\left(\frac{2\lambda_1}{\mu_1}\tau_1^2-3\right)+\frac{3}{2}(\tau_1^2-1)\right]\Omega_3+\frac{\mu_2(\mu_2-\mu_3)}{2\mu_1^2}\left[\left(\frac{b}{a}\right)^3-1\right]\left(\frac{2\lambda_2}{\mu_2}\tau_2^2-3\right)Z_3\\ &\left.+\left[\frac{\mu_2-\mu_3}{\mu_1}+\frac{2\mu_2+\mu_3}{\mu_1}\left(\frac{b}{a}\right)^3\right]\frac{\lambda_1}{2\mu_1}\left(\frac{b}{a}\right)^3\right]D_1^{-1}, \end{aligned} \tag{A.4}$$

$$\begin{aligned} p_{Ak}^{(1)} = p_{kA}^{(1)}= &\left[\frac{(2\mu_2+\mu_3)\mu_2}{2\mu_1^2}\left[\left(\frac{b}{a}\right)^3-\left(\frac{b}{a}\right)^6\right]-\frac{3}{2}\frac{\mu_2\mu_3}{\mu_1^2}\left(\frac{b}{a}\right)^3\right.\\ &-\frac{9}{10}\frac{\mu_2(\mu_2-\mu_3)}{\mu_1^2}(\tau_2^2-1)\cdot\left(\frac{b}{a}\right)^3\left[1-\left(\frac{b}{a}\right)^2\right]X_3\\ &+\left[\frac{\mu_2-\mu_3}{2\mu_1}+\frac{2\mu_2+\mu_3}{2\mu_1}\left(\frac{b}{a}\right)^3\right]\left[\frac{3}{2}-\frac{3}{5}\left(\frac{\lambda_1}{\mu_1}\tau_1^2+1\right)\right]\Omega_3\\ &+\frac{3\mu_2(\mu_2-\mu_3)}{20\mu_1^2}\left[\left[\left(\frac{b}{a}\right)^3-1\right]\left(3-\frac{2\lambda_2}{\mu_2}\tau_2^2\right)+(\tau_1^2-1)\right]Z_3\\ &\left.+\left[\frac{\mu_2-\mu_3}{2\mu_1}+\frac{2\mu_2+\mu_3}{2\mu_1}\left(\frac{b}{a}\right)^3\right]\left(\frac{b}{a}\right)^3\right]D_1^{-1}, \end{aligned} \tag{A.5}$$

$$p^{(1)}=-\frac{15}{10}(\tau_1^2-1)\Omega_3, \quad p^{(2)}=-\frac{15}{10}(\tau_2^2-1)Z_3, \tag{A.6}$$

$$q_{Ak}^{(2)}=q_{kA}^{(2)}=q_0^{(2)}-\frac{1}{2}(\tau_2^2-1)X_3,\quad q_{Ak}^{(3)}=q_{kA}^{(3)}=q_0^{(3)}-\frac{1}{2}(\tau_3^2-1)Y_3, \tag{A.7}$$

$$q^{(2)}=-5q_0^{(2)}+(\tau_2^2-1)Y_3,\quad q^{(3)}=-5q_0^{(3)}+(\tau_3^2-1)Y_3, \tag{A.8}$$

$$\begin{aligned}
t_0^{(1)} \;=\; & t_{Ak}^{(1)}=t_{kA}^{(1)}=-\frac{t^{(1)}}{5}=\left[-3\left[\frac{-D_2}{10}(\tau_2^2-1)\left(\frac{b}{a}\right)^2\right.\right.\\
&+\frac{2}{5}(\tau_2^2-1)\frac{(\mu_1-\mu_2)(4\mu_2+3\mu_3)}{\mu_1^2}\left[\left(\frac{b}{a}\right)^2-1\right]\cdot\left(\frac{b}{a}\right)^7\\
&+\left.\left[\frac{6(\mu_1-\mu_2)(\mu_2-\mu_3)}{5\mu_1^2}+\frac{(4\mu_1+3\mu_2)(4\mu_2+3\mu_3)}{10\mu_1^2}\left(\frac{b}{a}\right)^5\right]\left(\frac{b}{a}\right)^2(\tau_2^2-1)\right]Z_3\\
&-\frac{3}{10}(\tau_1^2-1)\left(\frac{b}{a}\right)^2D_2\Omega_3+\frac{\mu_2(4\mu_2+3\mu_3)}{10\mu_1^2}\left[1-\left(\frac{b}{a}\right)^7\right]\left(\frac{b}{a}\right)^7\left(3\tau_2^2\frac{\lambda_2}{\mu_2}-2\right)X_3\\
&\left.-\frac{7}{10}\frac{\mu_2\mu_3}{\mu_1^2}\left(\frac{b}{a}\right)^7\left(3\tau_3^2\frac{\lambda_3}{\mu_3}-2\right)Y_3\right]D_2^{-1},
\end{aligned} \tag{A.9}$$

$$\begin{aligned}
s_0^{(2)} \;=\; & \left[-\left[\frac{\mu_1-\mu_2}{\mu_1}+\frac{2\mu_2+\mu_3}{2\mu_1}\left(\frac{b}{a}\right)^3\right]\frac{\lambda_2}{\mu_1}\tau_2^2D_1X_1+\frac{(\mu_1-\mu_2)\lambda_3}{\mu_1^2}\tau_3^2D_1Y_1\right.\\
&+\frac{1}{5}(1-\tau_2^2)\cdot\left[\frac{2(\mu_1-\mu_2)(\mu_2-\mu_3)}{\mu_1^2}+\frac{(2\mu_1+\mu_2)(2\mu_2+\mu_3)}{\mu_1^2}\left(\frac{b}{a}\right)^5\right]X_3\\
&+\frac{2\mu_2+\mu_3}{10\mu_1}\left(\frac{2\lambda_1}{\mu_1}\tau_1^2-3\right)D_1\Omega_3-\frac{\mu_1+\mu_3}{10\mu_1}\left(\frac{2\lambda_2}{\mu_2}\tau_2^2-3\right)\frac{\mu_2}{\mu_1}Z_3\\
&\left.+\frac{\lambda_1(2\mu_2+\mu_3)}{2\mu_1^2}\left(\frac{b}{a}\right)^3\right]D_1^{-1},
\end{aligned} \tag{A.10}$$

$$\begin{aligned}
s_{Ak}^{(2)} \;=\; & 1-s_{kA}^{(2)}=\left[-\left[\frac{\mu_1-\mu_2}{\mu_1}+\frac{2\mu_2+\mu_3}{2\mu_1}\left(\frac{b}{a}\right)^3\right]\frac{\mu_2}{\mu_1}-\frac{(\mu_1-\mu_2)\mu_3}{\mu_1^2}\right.\\
&+\frac{2\mu_2+\mu_3}{2\mu_1}\cdot\left[\frac{3}{2}-\frac{3}{5}\left(\frac{\lambda_1}{\mu_1}\tau_1^2-1\right)\right]\Omega_3\\
&\left.-\frac{(\mu_2+\mu_3)}{\mu_1}\left[\frac{3}{2}-\frac{3}{5}\left(\frac{\lambda_2}{\mu_2}\tau_2^2-1\right)\right]\frac{\mu_2}{\mu_1}Z_3+\frac{5}{2}\frac{2\mu_2+\mu_3}{\mu_1}\left(\frac{b}{a}\right)^3\right]D_1^{-1},
\end{aligned} \tag{A.11}$$

$$\begin{aligned}
q_0^{(2)} \;=\; & \left[-\frac{6}{5}(\tau_2^2-1)\frac{(\mu_1-\mu_2)(\mu_2+3\mu_3)}{\mu_1^2}\left[\left(\frac{b}{a}\right)^2-1\right]Z_3+\frac{2(\mu_1-\mu_2)\mu_3}{5\mu_1^2}\left(\frac{3\lambda_3}{\mu_3}\tau_3^2-2\right)Y_3\right.\\
&\left.+\frac{1}{5}\left[-\left[\frac{4\mu_2+3\mu_3}{2\mu_1}\left(\frac{b}{a}\right)^7+2\frac{\mu_1-\mu_2}{\mu_1}\right]\frac{\mu_2}{\mu_1}\left(\frac{3\lambda_2}{\mu_2}\tau_2^2-2\right)+(\tau_2^2-1)D_2\right]X_3\right]D_2^{-1},
\end{aligned} \tag{A.12}$$

$$
\begin{aligned}
p_0^{(2)} &= \left[\frac{\lambda_2(2\mu_1+\mu_3)}{2\mu_1^2}\left(\frac{b}{a}\right)^3\tau_2^2 X_1+\frac{\lambda_3(2\mu_1+\mu_2)}{2\mu_1^2}\left(\frac{b}{a}\right)^3\tau_3^2 Y_1\right.\\
&\quad+\frac{(\mu_2-\mu_3)(2\mu_1+\mu_2)}{5\mu_1^2}(\tau_2^2-1)\cdot\left[\left(\frac{b}{a}\right)^3-\left(\frac{b}{a}\right)^5\right]X_3+\frac{\mu_2-\mu_3}{10\mu_1}\left(\frac{2\lambda_1}{\mu_1}\tau_1^2-3\right)\Omega_3\\
&\quad+\frac{1}{5}\left[\left[\frac{2\mu_1+\mu_2}{2\mu_1}\left(\frac{b}{a}\right)^3-\frac{\mu_2-\mu_3}{2\mu_1}\right]\cdot\frac{\mu_2}{\mu_1}\left(\frac{2\lambda_2}{\mu_2}\tau_2^2-3\right)+\frac{3}{2}(\tau_2^2-1)D_1\right]Z_3\\
&\quad\left.+\frac{\lambda_1(\mu_2-\mu_1)}{2\mu_1^2}\left(\frac{b}{a}\right)^3\right]D_1^{-1}, \qquad \text{(A.13)}
\end{aligned}
$$

$$
\begin{aligned}
p_{Ak}^{(2)} &= p_{kA}^{(2)}=\left[-\frac{(2\mu_1+\mu_3)\mu_2+(2\mu_1+\mu_2)\mu_3}{\mu_1^2}\left(\frac{b}{a}\right)^3\right.\\
&\quad-\frac{3(\mu_2-\mu_3)(2\mu_1+\mu_2)}{10\mu_1^2}(\tau^2-1)\left[1-\left(\frac{b}{a}\right)^2\right]\\
&\quad\cdot\left(\frac{b}{a}\right)^3 X_3+\frac{3}{10}\left[\left[\frac{2\mu_1+\mu_2}{2\mu_1}\left(\frac{b}{a}\right)^3-\frac{\mu_2-\mu_3}{2\mu_1}\right]\frac{\mu_2}{\mu_1}\left(3-\frac{2\lambda_2}{\mu_2}\tau_2^2\right)+D_1(\tau_2^2-1)\right]Z_3\\
&\quad\left.+\frac{3(\mu_2-\mu_3)}{20\mu_1}\left(3-\frac{2\lambda_1}{\mu_1}\tau_1^2\right)\Omega_3+\frac{5(\mu_2-\mu_3)}{2\mu_1}\left(\frac{b}{a}\right)^3\right]D_1^{-1}, \qquad \text{(A.14)}
\end{aligned}
$$

$$
\begin{aligned}
t_0^{(2)} &= t_{Ak}^{(2)}=t_{kA}^{(2)}=-\frac{t^{(2)}}{5}=\left[-3\left[\frac{6(\mu_1-\mu_2)(\mu_2-\mu_3)}{5\mu_1^2}\right.\right.\\
&\quad\left.+\frac{(4\mu_1+3\mu_2)(4\mu_2+3\mu_3)}{10\mu_1^2}\left(\frac{b}{a}\right)^5\right]\left(\frac{b}{a}\right)^2\cdot(\tau_2^2-1)Z_3\\
&\quad+\frac{\mu_2(4\mu_1+3\mu_3)}{10\mu_1^2}\left(\frac{b}{a}\right)^7\left(\frac{3\lambda_2}{\mu_2}\tau_2^2-2\right)X_3\\
&\quad\left.-\frac{1}{5}\left(2+\frac{3\mu_2}{2\mu_1}\right)\left(\frac{b}{a}\right)^7\frac{\mu_3}{\mu_1}\left(\frac{2\lambda_3}{\mu_3}\tau_3^2-2\right)Y_3\right]D_2^{-1}, \qquad \text{(A.15)}
\end{aligned}
$$

$$
\begin{aligned}
s_0^{(3)} &= \left[\left[(1-\tau_2^2)\left[\frac{2(\mu_1-\mu_2)(\mu_2-\mu_3)}{5\mu_1^2}+\frac{(2\mu_1+\mu_2)(2\mu_2+\mu_3)}{5\mu_1^2}\left(\frac{b}{a}\right)^5\right]\right.\right.\\
&\quad\left.+\frac{(\mu_2-\mu_3)(2\mu_1+\mu_2)}{5\mu_1^2}\cdot(\tau_2^2-1)\left[\left(\frac{b}{a}\right)^3-\left(\frac{b}{a}\right)^5\right]+\frac{1}{5}(\tau_2^2-1)D_1\right]X_3\\
&\quad-\frac{1}{5}(\tau_3^2-1)D_1Y_3+\frac{3\mu_2}{10\mu_1}\left(\frac{2\lambda_1}{\mu_1}\tau_1^2-3\right)\Omega_3\\
&\quad-\left[\frac{2\mu_1+\mu_2}{2\mu_1}\left[1-\left(\frac{b}{a}\right)^3\right]+\frac{\mu_2}{5\mu_1}\left(\frac{2\lambda_2}{\mu_2}\tau_2^2-3\right)D_1\right]Z_3\\
&\quad\left.-\frac{\lambda_2}{\mu_1}\tau_2^2X_1+\frac{\mu_1-\mu_2}{\mu_1}\left[1-\left(\frac{b}{a}\right)^3\right]\frac{\lambda_3}{\mu_1}\tau_3^2Y_1+\frac{3}{2}\frac{\lambda_1\mu_2}{\mu_1^2}\left(\frac{b}{a}\right)^3\right]D_1^{-1}, \qquad \text{(A.16)}
\end{aligned}
$$

$$
\begin{aligned}
s_{Ak}^{(3)} \;=\; 1-s_{kA}^{(3)} = &\left[\frac{\mu_2-\mu_1}{\mu_1}\left[1-\left(\frac{b}{a}\right)^3\right]\frac{\mu_2}{\mu_1}+\left[\frac{\mu_1-\mu_2}{\mu_1}-\frac{2\mu_1+\mu_2}{2\mu_1}\left(\frac{b}{a}\right)^3\right]\frac{\mu_3}{\mu_1}\right.\\
&-\frac{3}{10}\left[(1-\tau_2^2)\left[\frac{2(\mu_1-\mu_2)(\mu_2-\mu_3)}{\mu_1^2}+\frac{(2\mu_1+\mu_2)(2\mu_2+\mu_3)}{\mu_1^2}\left(\frac{b}{a}\right)^5\right]\right.\\
&\left.\left.+\frac{(\mu_2-\mu_3)(2\mu_1+\mu_2)}{\mu_1^2}\cdot(\tau_2^2-1)\left[\left(\frac{b}{a}\right)^3-\left(\frac{b}{a}\right)^5\right]+(\tau_2^2-1)D_1\right]X_3\\
&+\frac{3}{10}(\tau_3^2-1)D_1Y_3+\frac{9\mu_2}{20\mu_1}\left(3-\frac{2\lambda_1}{\mu_1}\tau_1^2\right)\Omega_3\\
&-\frac{3}{10}\left[\frac{2\mu_1+\mu_2}{2\mu_1}[1-\left(\frac{b}{a}\right)^3\right]\frac{\mu_2}{\mu_1}\left(3-\frac{2\lambda_2}{\mu_2}\tau_2^2\right)\Big]Z_3+\frac{3\mu_2}{2\mu_1}\left(\frac{b}{a}\right)^3\Big]D_1^{-1},
\end{aligned}
\tag{A.17}
$$

$$
\begin{aligned}
q_0^{(3)} \;=\; &\left[-3\left[\frac{2}{5}(\tau_2^2-1)\frac{(\mu_1-\mu_2)(4\mu_2+3\mu_3)}{\mu_1^2}\left[\left(\frac{b}{a}\right)^2-1\right]+\left[\frac{6(\mu_1-\mu_2)(\mu_2-\mu_3)}{\mu_1^2}\right.\right.\right.\\
&\left.\left.+\frac{4(\mu_1+3\mu_2)(4\mu_2+3\mu_3)}{10\mu_1^2}\left(\frac{b}{a}\right)^5\right]\left(\frac{b}{a}\right)^2(\tau_2^2-1)-\frac{1}{10}(\tau_2^2-1)D_2\right]Z_3\\
&-\left[2\frac{\mu_1-\mu_2}{\mu_1}\left[1-\left(\frac{b}{a}\right)^7\right]\frac{\mu_2}{5\mu_1}\left(\frac{3\lambda_2}{\mu_2}\tau_2^2-2\right)\right]X_3\\
&+\frac{1}{5}\left[\frac{2(\mu_1-\mu_2)}{\mu_1}-\left(2+\frac{3\mu_2}{2\mu_1}\right)\left(\frac{b}{a}\right)^7\right.\\
&\left.\left.\cdot\frac{\mu_3}{\mu_1}\left(\frac{3\lambda_3}{\mu_3}\tau_3^2-2\right)+(\tau_3^2-1)\right]D_2Y_3\right]D_2^{-1},
\end{aligned}
\tag{A.18}
$$

where

$$
D_1=\frac{2(\mu_2-\mu_3)(\mu_1-\mu_2)}{\mu_1^2}+\frac{(2\mu_2+\mu_3)(2\mu_1+\mu_2)}{\mu_1^2}\left(\frac{b}{a}\right)^3, \tag{A.19}
$$

$$
D_2=\frac{12(\mu_2-\mu_3)(\mu_1-\mu_2)}{\mu_1^2}+\frac{(4\mu_2+3\mu_3)(4\mu_1+3\mu_2)}{\mu_1^2}\left(\frac{b}{a}\right)^7, \tag{A.20}
$$

and (X_1,Y_1), (X_3,Y_3,Z_3,Ω_3) are the solutions, of the following systems

$$
\begin{aligned}
&\left(\tau_2^2D_1+3\left[\frac{\mu_1-\mu_2}{\mu_1}+\frac{2\mu_2+\mu_3}{2\mu_1}\left(\frac{b}{a}\right)^3\right]\tau_2^2\frac{\lambda_2}{\mu_1}\right)X_1\\
&\quad-3\tau_3^2\frac{(2\mu_2+\mu_3)\lambda_3}{2\mu_1^2}Y_1=3\frac{2\mu_2+\mu_3}{2\tau_1^2\mu_1}\left(\frac{b}{a}\right)^3\mathbf{a}_1\cdot\hat{\mathbf{k}},
\end{aligned}
\tag{A.21}
$$

$$
\begin{aligned}
&-3\tau_2^2\frac{\lambda_2(\mu_1-\mu_2)}{\mu_1^2}\frac{b^3-a^3}{a^3}X_1+\tau_3^2\left(1+3\frac{\lambda_3(2\mu_1+\mu_2)}{2\mu_1^2}\left(\frac{b}{a}\right)^3\right)Y_1\\
&\quad=\frac{9\mu_2}{2\tau_1^2\mu_1}\left(\frac{b}{a}\right)^3\mathbf{a}_1\cdot\hat{\mathbf{k}},
\end{aligned}
\tag{A.22}
$$

$$\left[\left(1+\frac{3}{5}(\tau_2^2-1)\right)D_2-\frac{7}{5}\frac{\mu_2}{\mu_1}\left(\frac{3\lambda_2}{\mu_2}\tau_2^2-2\right)\left[\frac{4\mu_2+3\mu_3}{2\mu_1}\left(\frac{b}{a}\right)^7+2\frac{\mu_1-\mu_2}{\mu_1}\right]\right]X_3$$
$$+\frac{42}{5}(\tau_2^2-1)\frac{(\mu_1-\mu_2)(4\mu_2+3\mu_3)}{\mu_1^2}\left[\left(\frac{b}{a}\right)^2-1\right]\left(\frac{r}{a}\right)^5 Z_3$$
$$+\frac{14}{5}\left(2-\frac{3\lambda_3}{\mu_3}\tau_3^2\right)\frac{\mu_3(\mu_1-\mu_2)}{\mu_1^2}Y_3=0, \tag{A.23}$$

$$-\frac{\mu_2}{\mu_1}\left(2-\frac{3\lambda_2}{\mu_2}\tau_2^2\right)\frac{2(\mu_1-\mu_2)}{2\mu_1}\left[1-\left(\frac{b}{a}\right)^7\right]X_3$$
$$+21\left[\frac{2}{5}(\tau_2^2-1)\frac{(\mu_1-\mu_2)(4\mu_2+3\mu_3)}{\mu_1^2}\left[\left(\frac{b}{a}\right)^2-1\right]\right.$$
$$+\left[\frac{6(\mu_1-\mu_2)(\mu_2-\mu_3)}{5\mu_1^2}+\frac{(4\mu_1+3\mu_2)(4\mu_2+3\mu_3)}{10\mu_1^2}\left(\frac{b}{a}\right)^5\right]\left(\frac{b}{a}\right)^2(\tau_2^2-1)$$
$$\left.-\frac{1}{10}(\tau_2^2-1)D_2\right]\left(\frac{r}{a}\right)^5 Z_3$$
$$+\left[1+\frac{3}{5}(\tau_3^2-1)-\frac{14}{5}\frac{\mu_3}{\mu_1}\left(\frac{3\lambda_3}{\mu_3}\tau_3^2-2\right)+2\frac{\mu_1-\mu_2}{\mu_1}-\left(2+\frac{3\mu_2}{2\mu_1}\right)\left(\frac{b}{a}\right)^7\right]Y_3=0, \tag{A.24}$$

$$\frac{3(\mu_2-\mu_3)(2\mu_1+\mu_2)}{5\mu_1^2}(\tau_2^2-1)\left[\left(\frac{b}{a}\right)^3-\left(\frac{b}{a}\right)^5\right]\left(\frac{a}{r}\right)^5 X_3$$
$$+\left[\left(1+\frac{2}{5}(\tau_2^2-1)\right)D_1+\frac{3\mu_2}{5\mu_1}\left(3-\frac{2\lambda_2}{\mu_2}\tau_2^2\right)\right.$$
$$\left.\cdot\left[\frac{2\mu_1+\mu_2}{2\mu_1}\left(\frac{b}{a}\right)^3-\frac{\mu_2-\mu_3}{2\mu_1}\right]Z_3-\frac{3}{5}\left(3-\frac{2\lambda_1}{\mu_1}\tau_1^2\right)\frac{\mu_2-\mu_3}{2\mu_1}\Omega_3\right.$$
$$=3\frac{\mu_2-\mu_3}{\mu_1}\left(\frac{a}{r}\right)^3[\mathbf{a}_1\cdot\hat{\mathbf{k}}-3(\mathbf{a}_1\otimes\hat{\mathbf{k}}):(\hat{\mathbf{r}}\otimes\hat{\mathbf{r}})], \tag{A.25}$$

$$\frac{9}{5}\frac{\mu_2(\mu_2-\mu_3)}{\mu_1^2}(\tau_2^2-1)\left(\frac{b}{a}\right)^3\left[1-\left(\frac{b}{a}\right)^2\right]\left(\frac{a}{r}\right)^5 X_3+\left[\left(1+\frac{2}{5}(\tau_1^2-1)\right)D_1\right.$$
$$\left.-\frac{3}{10}\left(3-\frac{2\lambda_1}{\mu_1}\tau_1^2\right)\left[\frac{\mu_2-\mu_3}{\mu_1}+\frac{2\mu_2+\mu_3}{\mu_1}\left(\frac{b}{a}\right)^3\right]\right]\Omega_3$$
$$+\frac{3}{10}\frac{\mu_2}{\mu_1}\left(\frac{2\lambda_2}{\mu_2}\tau_2^2-3\right)\frac{\mu_2-\mu_3}{\mu_1}\left[\left(\frac{b}{a}\right)^3-1\right]Z_3$$
$$=\left[\frac{(\mu_1+2\mu_2)(\mu_2-\mu_3)}{\mu_1^2}+\frac{(\mu_1-\mu_2)(2\mu_2+\mu_3)}{\mu_1^2}\left(\frac{b}{a}\right)^3\right]$$
$$\cdot\left(\frac{b}{a}\right)^3\left(\frac{a}{r}\right)^3[\mathbf{a}_1\cdot\hat{\mathbf{k}}-3(\mathbf{a}_1\otimes\hat{\mathbf{k}}):(\hat{\mathbf{r}}\otimes\hat{\mathbf{r}})]. \tag{A.26}$$

Acknowledgments

This work is partially supported by the Greek Secretariat General for Research and Technology.

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